Fundamental theorem of Algebra. B.Sc Part I (Honrs) Group (C). (Theory of Equations)

Variable:- A symbol which can take number of values is known as variable.

Polynomial - The function of n^{th} degree where n being assumed to be positive is said to be Polynomial

i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$.

Equation - For some values of x if two polynomials become equal ~~from~~ then this relation is known as equation and the values of x obtained from given equation are known as roots of the given equation.

i.e. $x^2 - 5x + 6 = 0$ is a equation and the value of x i.e. 2, 3 are said to be roots of the given equation.

Numerical equation - A function $f(x)$ is said to be numerical equation when the coefficient of various powers of x are numerical.

i.e. $f(x) = 7x^3 + 3x^2 + 2x + 3 = 0$ is a numerical equation.

Theorem 1. State and prove fundamental theorem of Algebra.

Statement:- Every polynomial equation with real coefficients has at least one root.

$$\text{let } f(x) = a_0 + a_1 x + a_2 x^2 + \text{---} + a_n x^n,$$

Where $n \geq 1$ and $a_2 \neq 0$. So that $f(x)$ is a polynomial of degree one or greater. Then the equation $f(x) = 0$ has at least one root.

Proof:- Let if possible $f(x) \neq 0$, for any value of x.

Then the function $\phi(x) = \frac{1}{f(x)}$ exists.

$$\text{Let } \phi(x) = \frac{1}{\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \text{---} + \frac{a_{n-1}}{x}}$$

$$\to 0 \text{ as } x \to \infty.$$

So for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|\phi(x)| < \epsilon$ for $|x| > \delta$

Here clearly $\phi(x)$ is continuous in the bounded closed domain $|x| < \delta$. Then we must have $\phi(x)$ is bounded in the closed domain $|x| < \delta$.

So there exists a number l such that

$$|\phi(x)| < l \text{ for } |x| \leq \delta$$

let $M = \text{Max}(\epsilon, l)$, then we have

$$|\phi(x)| = \left|\frac{1}{f(x)}\right| < M \text{ for every } x$$

Then by Liouvillis theorem $f(x)$ is constant which leads contradiction to our supposition, because $f(x)$ is not a constant when $n = 1, 2, 3, \text{---}$ and $a_n \neq 0$.

i.e. $f(x) = 0$ for at least one value of x.

So the equation $f(x) = 0$ must have at least one root.

Proved.

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